Complement on the proof of CHSH inequality.

In the previous proof given in the notes we assumed for simplicity that the outcome of the local measurement of Alice and Bob are <u>deterministic</u> functions $A(\alpha, \lambda)$ $A(\alpha', \lambda)$ $B(\beta, \lambda)$ $B(\beta', \lambda)$ of the meas settings $\alpha, \alpha', \beta, \beta'$ and of the "hidden variable" or "state" characterizing the pair distributed by the source. The proof was then based on the remark that if these fcts take values in $\{\pm 1\}$ then

$$|A(\alpha\lambda)B(\beta\lambda) - A(\alpha\lambda)B(\beta'\lambda) + A(\alpha'\lambda)B(\beta\lambda)$$
$$+ A(\alpha'\lambda)B(\beta'\lambda)| \leq 2$$

and then $\int d\lambda\, h(\lambda)\, |\ldots| \leq 2$ also.

Now we give a proof in a more general situation where we assume that the outcomes of the local measurements are <u>random variables</u>

a, b distributed according to the conditional distribution:

$p(a, b \mid \alpha, \beta, \lambda)$ = probability that Alice finds $a \in \{\pm 1\}$, Bob finds $b \in \{\pm 1\}$ given that the meas devices are set to α and β and the "hidden variable" or state of the pair is λ.

<u>We make a locality assumption:</u>

$$p(a, b \mid \alpha, \beta, \lambda) = p_1(a \mid \alpha, \lambda)\, p_2(b \mid \beta, \lambda)$$

which means that the result of Alice cannot be influenced by Bob's setting & measurement outcome. And vice versa.

<u>Theorem.</u> Under the locality assumption

$$\underbrace{\text{Correlation coefficient}}_{S_{theory}} \equiv Av(ab) - Av(ab') + Av(a'b) + Av(a'b')$$

satisfies $|S_{theory}| \leq 2$.

(Bell-CHSH inequality)

<u>Proof</u>

$$Av(ab) = \int d\lambda\, h(\lambda) \sum_{a,b} ab\, p(a,b|\alpha,\beta,\lambda)$$

$$Av(ab') = \int d\lambda\, h(\lambda) \sum_{a\,b'} ab'\, p(a,b'|\alpha,\beta',\lambda)$$

$$Av(a'b) = \int d\lambda\, h(\lambda) \sum_{a'\,b} a'b\, p(a',b|\alpha',\beta,\lambda)$$

$$Av(a'b') = \int d\lambda\, h(\lambda) \sum_{a'\,b'} a'b'\, p(a',b'|\alpha',\beta',\lambda)$$

Remark: of course here a, b, a', b' are dummy variables.
but the result depends on $\alpha, \beta, \alpha', \beta'$ ← not dummy

Because of the locality assumption we have

$$p(a, b / \alpha, \beta, \lambda) = p_1(a/\alpha\lambda)\, p_2(b/\beta\lambda)$$

$$p(a, b'/\alpha\beta'\lambda) = p_1(a/\alpha\lambda)\, p_2(b'/\beta'\lambda)$$

$$p(a'b/\alpha'\beta\lambda) = p_1(a'/\alpha'\lambda)\, p_2(b/\beta\lambda)$$

$$p(a'b'/\alpha'\beta'\lambda) = p_1(a'/\alpha'\lambda)\, p_2(b'/\beta'\lambda).$$

and these can be viewed as 4 marginal distributions of the "global" distribution:

$$Q(a, a', b, b') = p_1(a/\alpha\lambda)\, p_1(a'/\alpha'\lambda)\, p_2(b/\beta\lambda)\, p_2(b'/\beta'\lambda)$$

Note that $Q(a, a', b, b')$ depends on $\alpha, \alpha', \beta, \beta', \lambda$ but this dependence will not play a role so is not indicated explicitly.

Indeed for example:

$$\sum_{a'b'} Q(a,a',b,b') = p_1(a|\alpha\lambda)\, p_2(b|\beta\lambda)$$

$$\cdot \underbrace{\sum_{a'} p_1(a'|\alpha'\lambda)}_{1} \underbrace{\sum_{b'} p_2(b'|\beta'\lambda)}_{1}$$

$$= p(ab|\alpha\beta\lambda).$$

idem for other cases.

This implies we can write

$$Av(ab) = \int d\lambda\, h(\lambda) \sum_{a,a',b,b'} ab\, Q(a,a',b,b')$$

$$Av(ab') = \int d\lambda\, h(\lambda) \sum_{aa'bb'} ab'\, Q(a,a',b,b')$$

$$Av(a'b) = \int d\lambda\, h(\lambda) \sum_{aa'bb'} a'b\, Q(a,a',b,b')$$

$$Av(a'b') = \int d\lambda\, h(\lambda) \sum_{aa'bb'} a'b'\, Q(a,a',b,b')$$

and therefore:

$$S = Av(ab) - Av(ab') + Av(a'b) + Av(a'b')$$

$$= \int d\lambda\, h(\lambda) \sum_{aa'bb'} Q(a,a',b,b')\{ab - ab' + a'b + a'b'\}$$

Now since $a, a', b, b' \in \{\pm 1\}$ we have as before:

$$ab - ab' + a'b + a'b' = a(b-b') + a'(b+b')$$
$$\in \{-2, +2\}.$$

$\Rightarrow$ By triangle inequality:

$$|S| \leq \int d\lambda\, h(\lambda) \sum_{aa'bb'} Q(aa'bb') \underbrace{|\{\cdots\}|}_{\leq 2}$$

$$= 2 \int d\lambda\, h(\lambda) \underbrace{\sum_{aa'bb'} Q(aa'bb')}_{1 \text{ because it's a prob distr.}}$$

$$= 2 \underbrace{\int d\lambda\, h(\lambda)}_{1} = 2.$$

∎

Discussion of the locality assumption.

The whole proof rests on the locality assumption

$$p(a,b|\alpha,\beta,\lambda) = p_1(a|\alpha,\lambda)\, p_2(b|\beta,\lambda)$$

In QM this is precisely what fails if $|\psi\rangle$ is entangled!. According to QM:
(Born rule)

$$\begin{cases} p(a=1, b=1/\alpha,\beta,\lambda) = |\langle\alpha,\beta|\psi\rangle|^2 \\ p(a=1, b=-1/\alpha,\beta,\lambda) = |\langle\alpha,\beta_\perp|\psi\rangle|^2 \\ p(a=-1, b=1/\alpha,\beta,\lambda) = |\langle\alpha_\perp,\beta|\psi\rangle|^2 \\ p(a=-1, b=-1/\alpha,\beta,\lambda) = |\langle\alpha_\perp,\beta_\perp|\psi\rangle|^2 \end{cases}$$

For $|\psi\rangle = |\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ then probabilities can be computed and we find a formula which cannot be factorized as in the locality assumption (good exercise).

Suppose now that instead of the Bell state $|\psi^+\rangle$ we take a <u>product state</u>:

$$|\psi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$$

So the qubits of Alice and Bob have a "precise" identity: $|\varphi_A\rangle$ and $|\varphi_B\rangle$.

We have by the previous Born rule (here "$\lambda \approx |\psi\rangle$").

$$p(1, 1 | \alpha \beta \lambda) = |\langle\alpha|\varphi_A\rangle|^2 \cdot |\langle\beta|\varphi_B\rangle|^2$$

$$p(1, -1 | \alpha \beta \lambda) = |\langle\alpha|\varphi_A\rangle|^2 \cdot |\langle\beta_\perp|\varphi_B\rangle|^2$$

$$p(-1, 1 | \alpha \beta \lambda) = |\langle\alpha_\perp|\varphi_A\rangle|^2 \cdot |\langle\beta|\varphi_B\rangle|^2$$

$$p(-1, -1 | \alpha \beta \lambda) = |\langle\alpha_\perp|\varphi_A\rangle|^2 \, |\langle\beta_\perp|\varphi_B\rangle|^2 .$$

This is a factorized distribution. More precisely define the notation $\alpha_a = \begin{cases} \alpha, & a=0 \\ \alpha_\perp & a=1 \end{cases}$

$$\beta_b = \begin{cases} \beta, & b=0 \\ \beta_\perp, & b=1 \end{cases}$$

Then

$$p(a,b|\alpha\beta\lambda) = \underbrace{|\langle \alpha_a | \varphi_A\rangle|^2}_{p_1(a|\alpha,\lambda)} \cdot \underbrace{|\langle \beta_b | \varphi_B\rangle|^2}_{p_2(b|\beta,\lambda)}.$$

(Here "$\lambda \equiv |\varphi_A\rangle \otimes |\varphi_B\rangle$" describes state given by source.).

Corollary: For a product state we have

$$|\langle \varphi_A \otimes \varphi_B | A\otimes B - A\otimes B' + A'\otimes B + A'\otimes B' | \varphi_A \otimes \varphi_B\rangle| \leq 2.$$

like for the classical Bell-CHSH inequality.

#.